

Smooth, global PL functions are non-linear least squares

July 17, 2026 – FoCM

Chris Criscitiello

with

Nicolas Boumal, Quentin Rebjock

NSF award DMS-
2609677

UPenn



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arXiv: <https://arxiv.org/pdf/2604.07972>

Race to the bottom: <https://www.racetothetbottom.xyz/posts/global-polyak-lojasiewicz/>

UPenn



PL condition

$$\min_{x \in \mathbb{R}^n} f(x)$$

PŁ condition

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Function classes:

- Convex functions
- Smooth, μ -strongly convex functions
- Functions with L -Lipschitz gradient
- ...

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- **This talk:** smooth, globally PŁ functions

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μ -globally PŁ: $\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*), \quad \forall x \in \mathbb{R}^n, \quad \mu > 0$

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Polyak '63, Łojasiewicz '65, Karimi et al. '16, Liu et al. '20

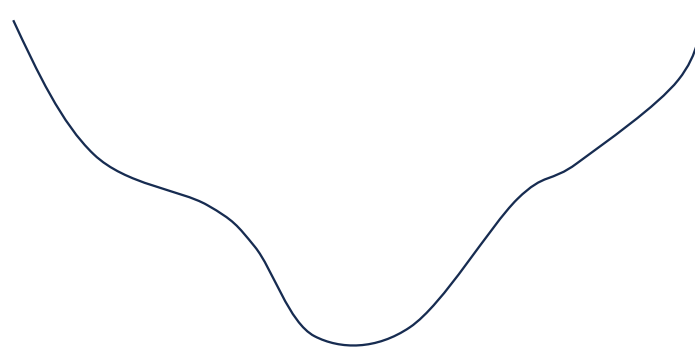
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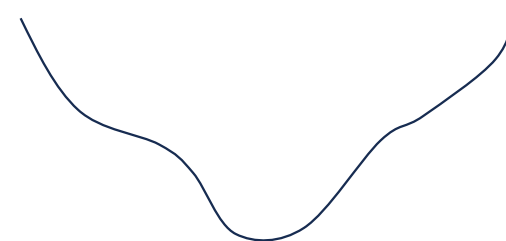
μ -strongly convex functions are μ -globally PL.

But the converse is not true: f need not be convex!

Why care about PL?

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*), \quad \forall x \in \mathbb{R}^n$$

Nonconvex



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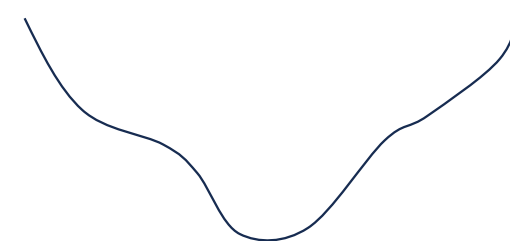
Nonconvex

Gradient descent/flow behaves nicely

L -Lipschitz gradient and μ -PŁ imply **linear convergence** (short proof!)

$$f(x_{k+1}) - f^* \leq \left(1 - \frac{\mu}{L}\right) (f(x_k) - f^*)$$

$$x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k)$$



Why care about PŁ?

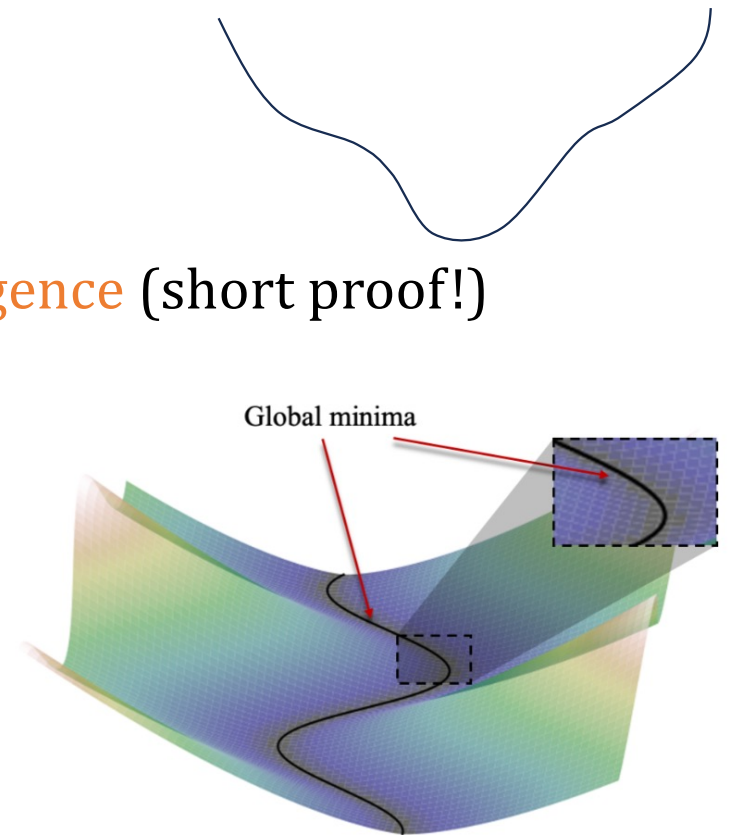
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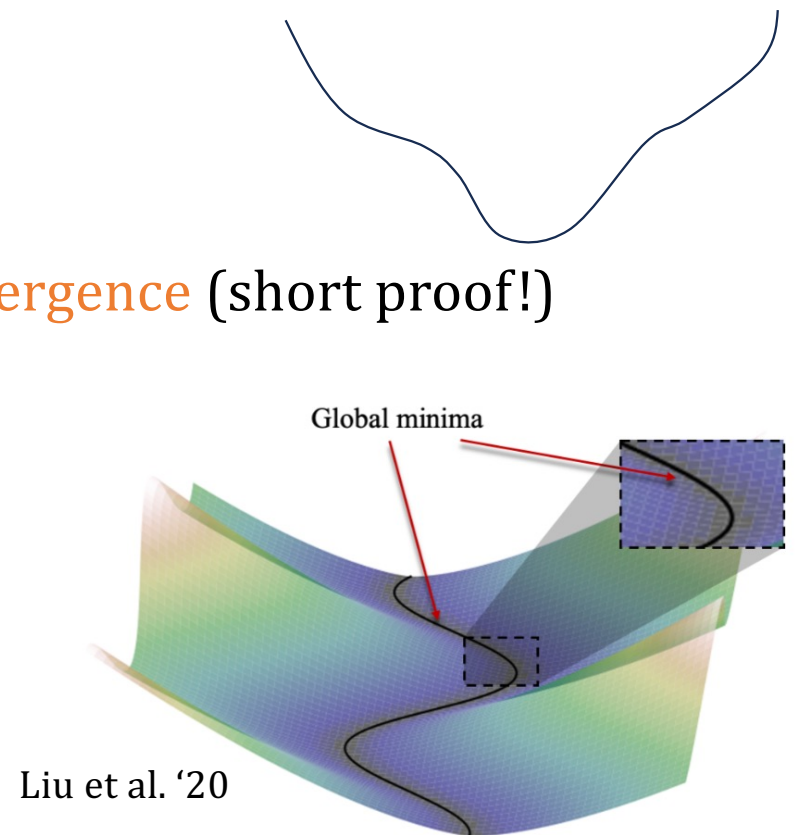
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Applications: control (Fazel et al. '18),
statistics (Chewi et al. '20)



Example

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*), \quad \forall x \in \mathbb{R}^n$$

Example: non-linear least squares

$$f(x) = \|F(x) - b\|^2 \quad F: \mathbb{R}^n \rightarrow \mathbb{R}^k$$

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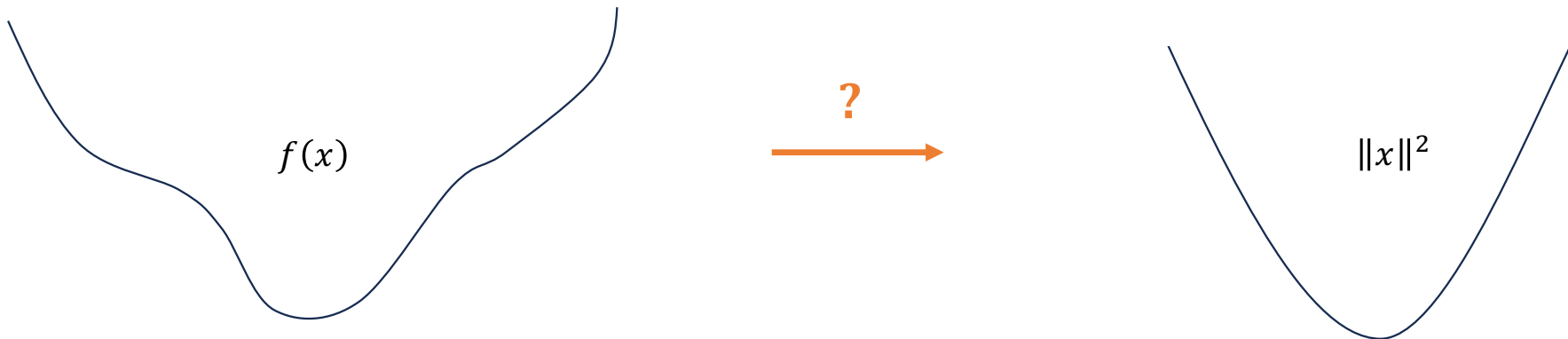
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We show all globally PL functions are of this form!

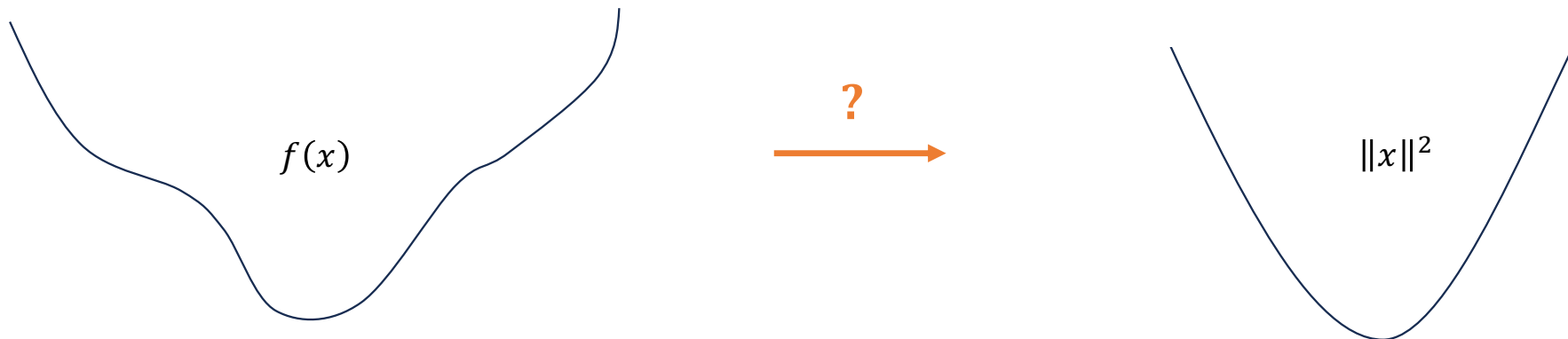
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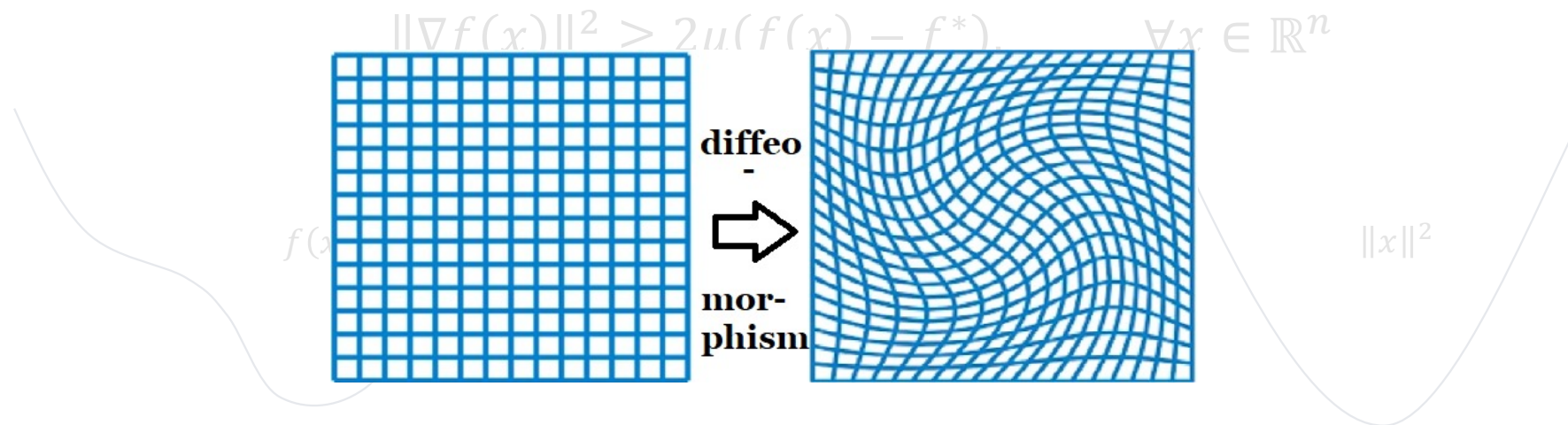
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If f is smooth and globally PL with **unique minimizer**, does there exist a **diffeomorphism** $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$f(x) = f^* + \|\phi(x)\|^2 \quad ?$$

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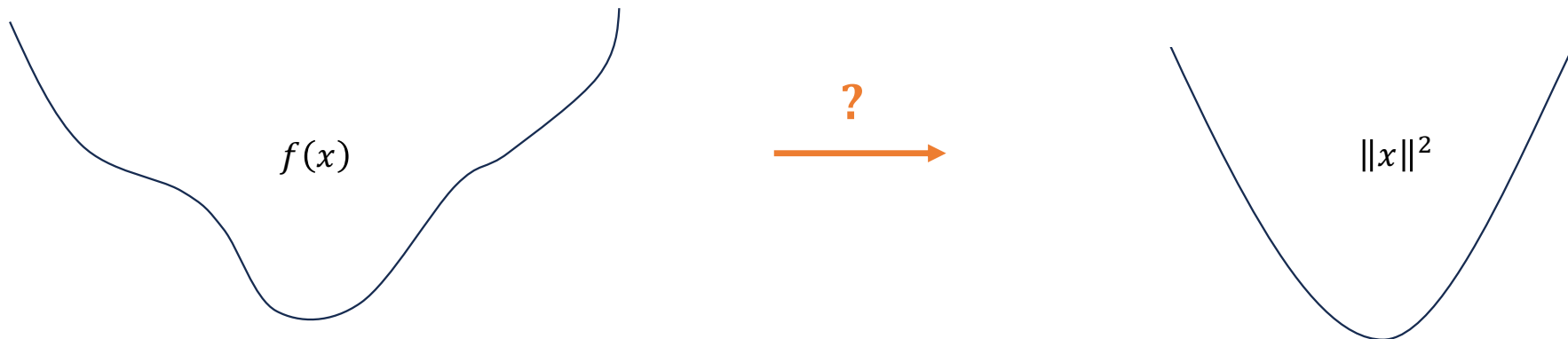


If A diffeomorphism is a smooth map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ with smooth inverse ϕ a diffeomorphism $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

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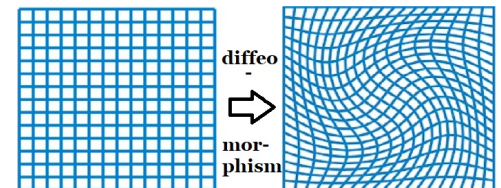
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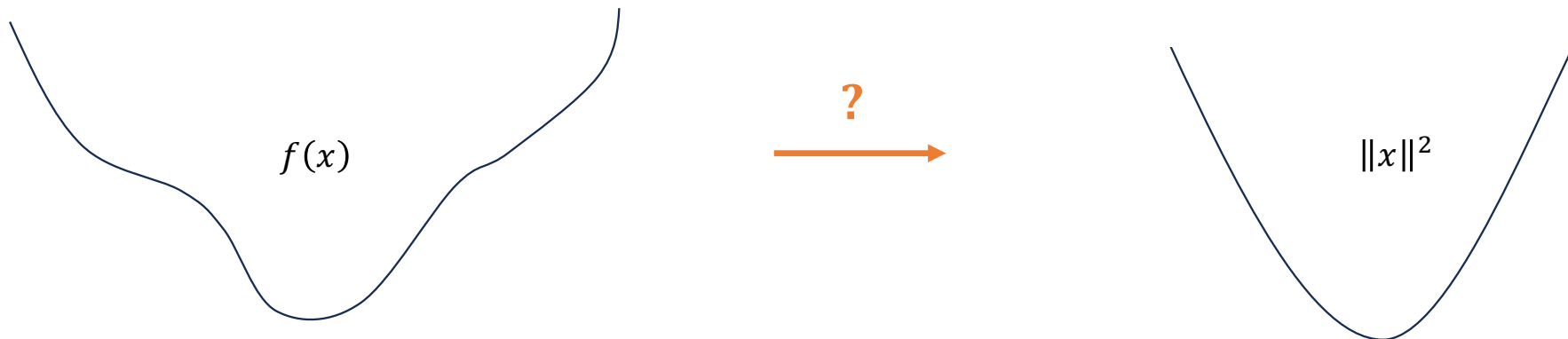
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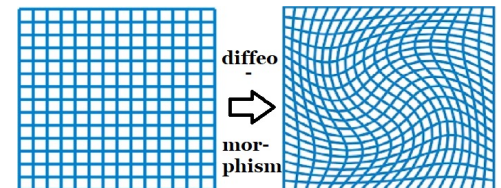
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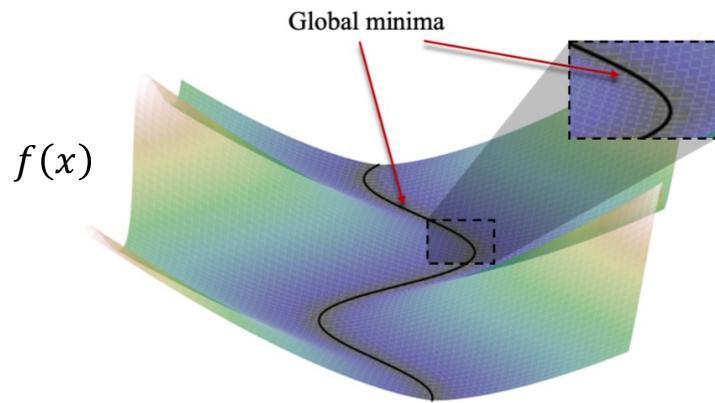
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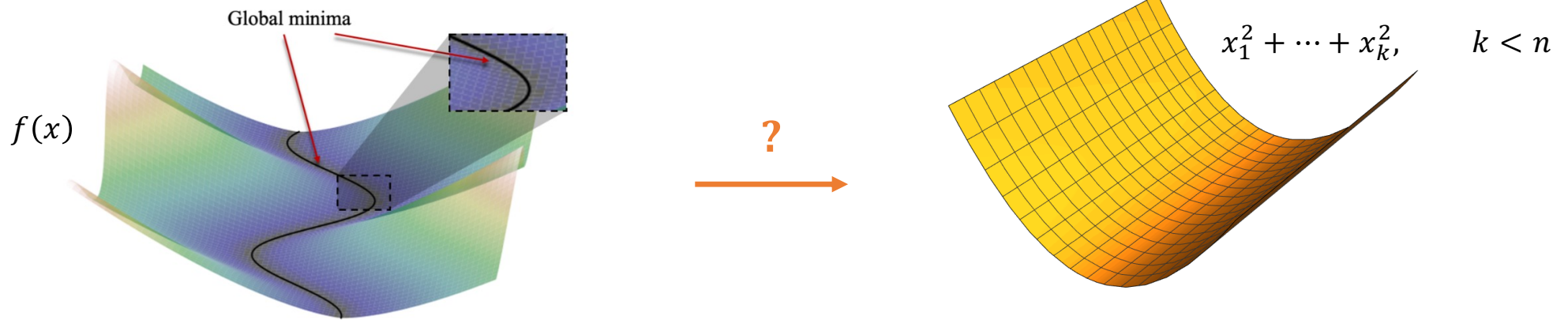


Baby Thm: **Yes**

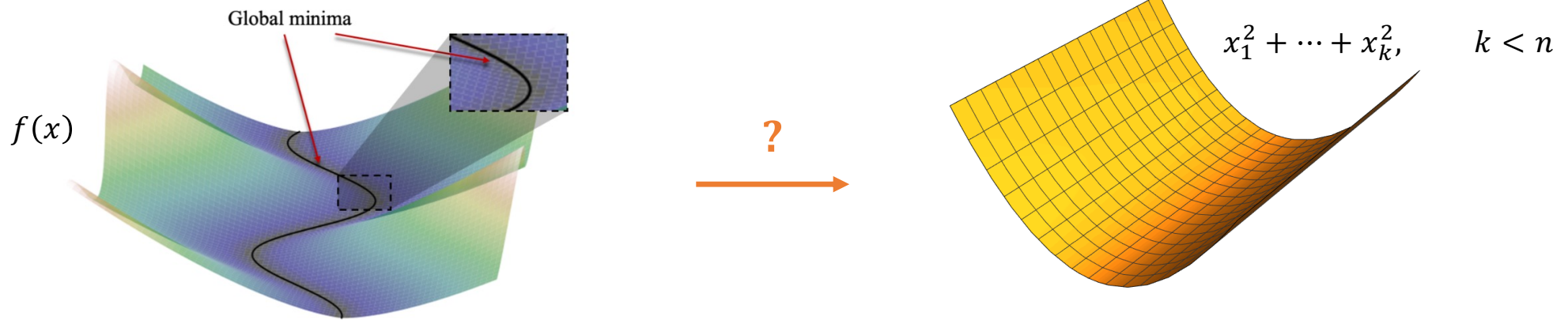
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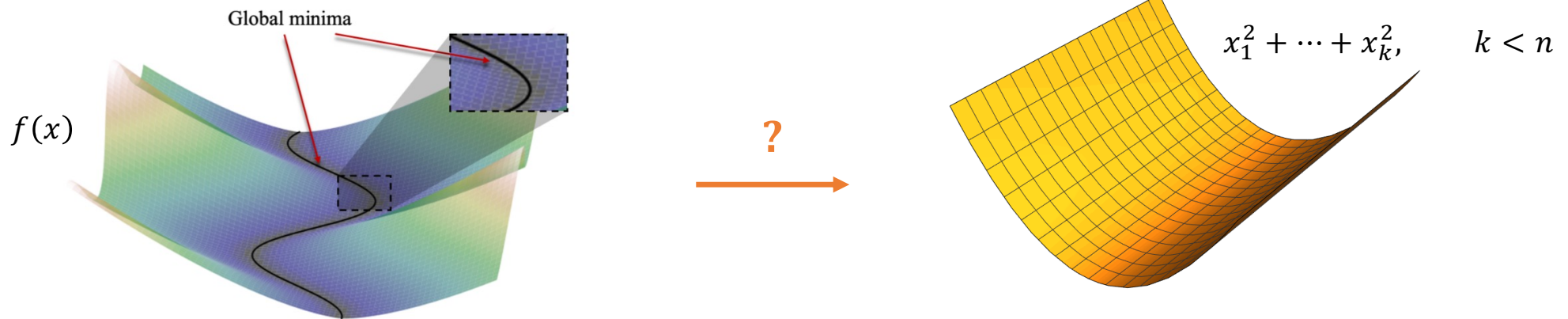
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$$S = \{x : \nabla f(x) = 0\} = \{x : f(x) = f^*\},$$

Short answer: **No** in general. Yes if $S \cong \mathbb{R}^{n-k}$.

Main results

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S cannot be a circle, sphere, cylinder, ...

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Implies $\phi(S) = S \times \{0\}$. Converse:

Thm (Building PL fcts): Let S be a submanifold of $\mathbb{R}^n$. If there exists a diffeomorphism

$$\phi : \mathbb{R}^n \rightarrow S \times \mathbb{R}^k \text{ with } \phi(S) = S \times \{0\}$$

then S is the argmin set of a globally PL function.

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Cor (Characterization of S): Let S be a smooth manifold (without boundary).

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Certainly includes $\mathbb{R}^m$. Any others?

Doesn't include circle, sphere, closed disk, cross, ...

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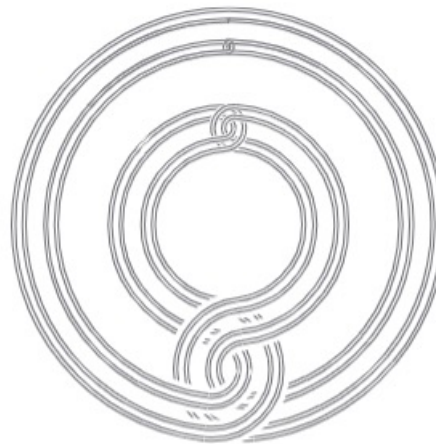
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Famous Whitehead manifold

Calegari '19

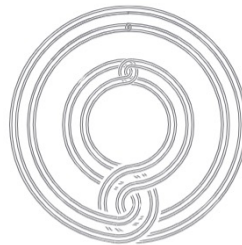
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Consequence: There exists a smooth, globally PL function f whose minimizer set S is diffeomorphic to a Whitehead manifold, and in particular is not homeomorphic to $\mathbb{R}^m$.

What if S isn't crazy?

Cor: If S is diffeomorphic to $\mathbb{R}^m$ (e.g., if $\dim S \leq 2$), then there exists a diffeomorphism $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

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Summary:

Either f is simple, i.e., $f(\phi^{-1}(x)) = x_1^2 + \cdots + x_k^2$

Or S has to be wild, i.e., smooth and contractible but not diffeo to $\mathbb{R}^m$

Punchlines + Questions

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Questions:

C^p regularity?

Make results quantitative?

Possible embeddings of $S \subset \mathbb{R}^n$?

Appendix

Remainder of talk

Baby Thm: If f is smooth and globally PL with **unique minimizer**, there exists a **diffeomorphism** $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

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Next: Some geometric intuition behind proofs

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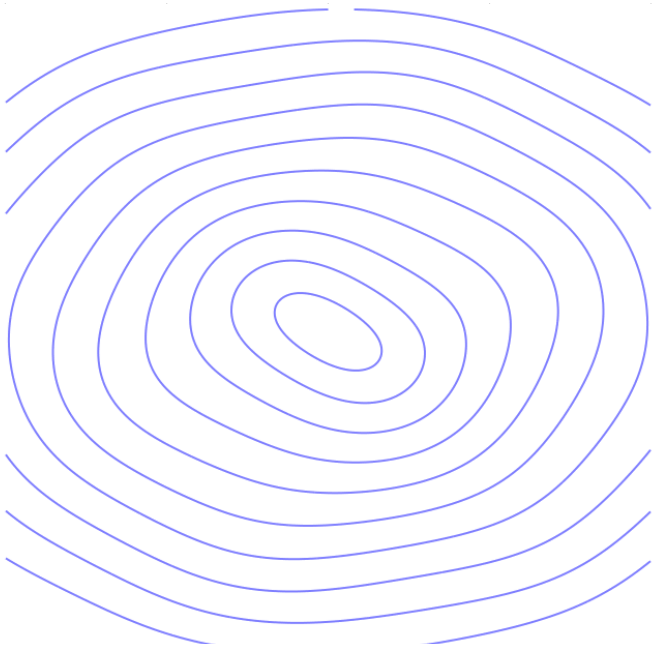
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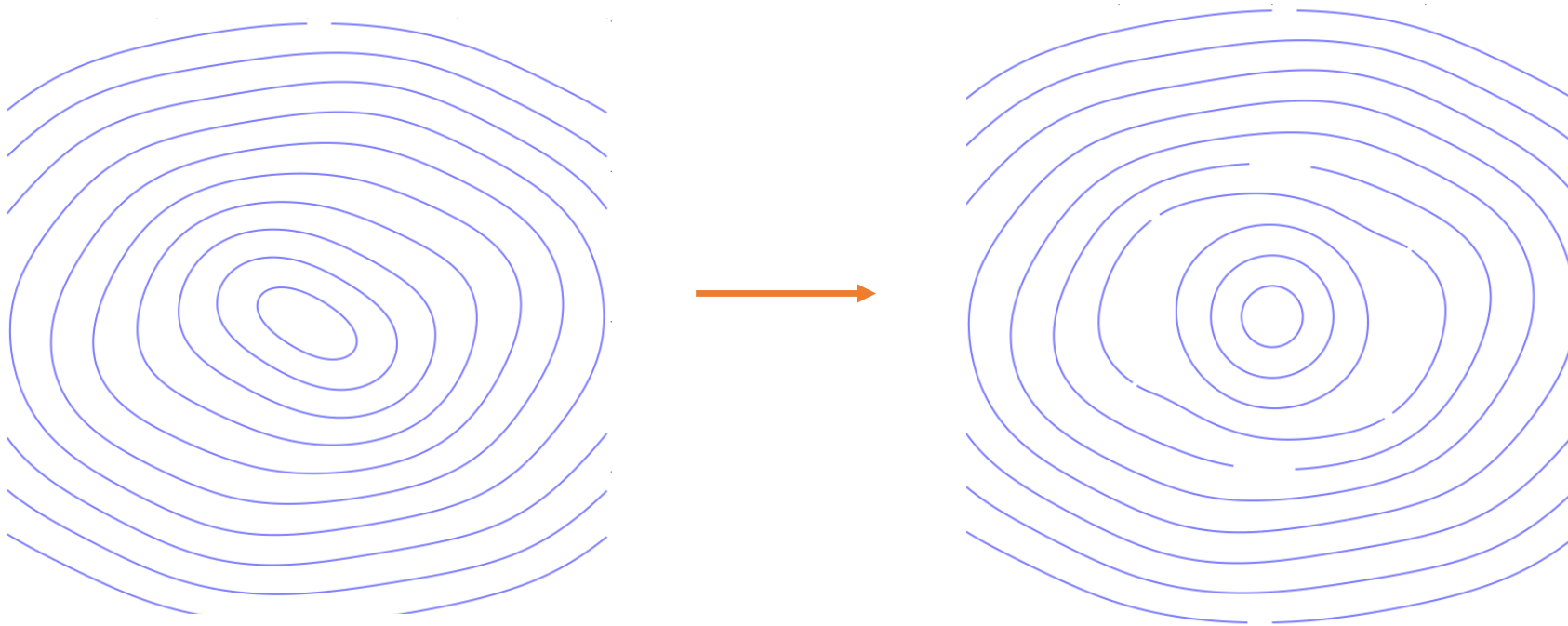


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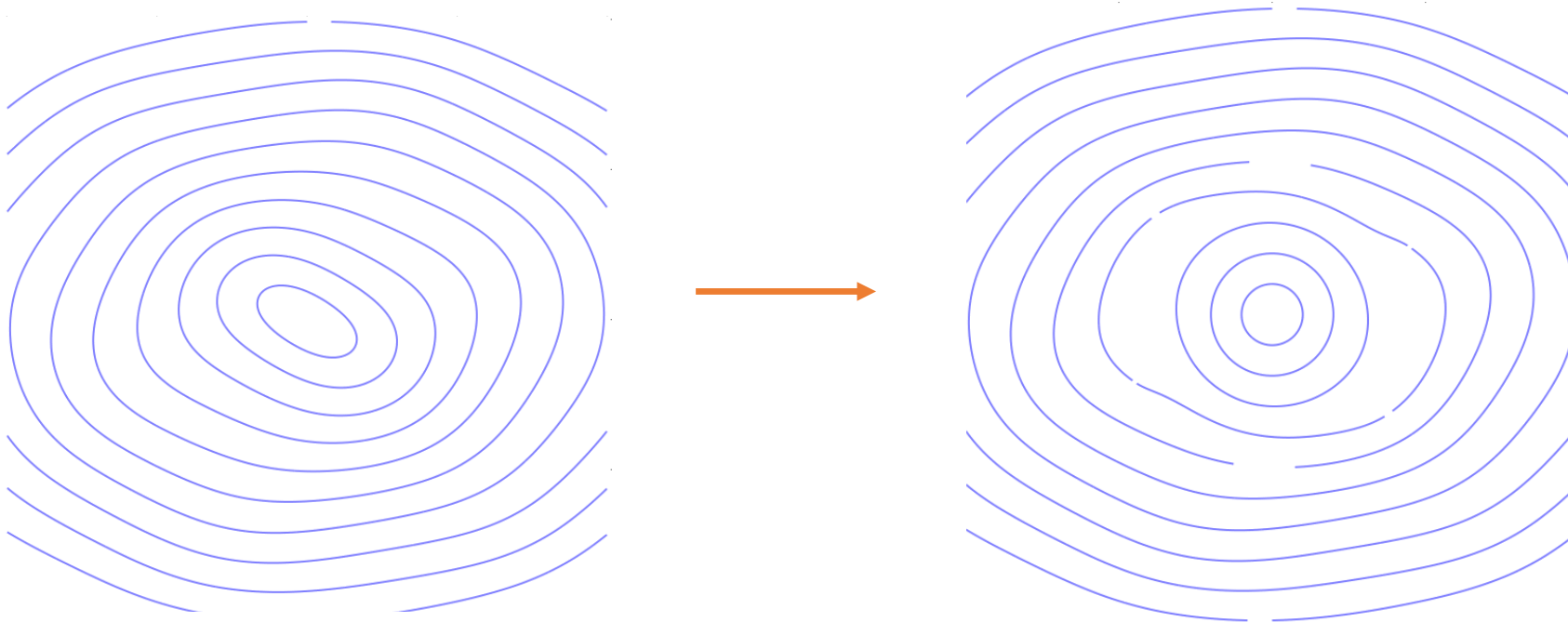
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Baby Thm: If f is smooth and globally PL with **unique minimizer**, there exists a **diffeomorphism** $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$f(x) = \|\phi(x)\|^2$$

Step 1: Make f locally a squared distance (**Morse Lemma**)

Step 2: Do it globally with grad flow $x'(t) = -\nabla f(x(t))$, $x(0) = x_0$

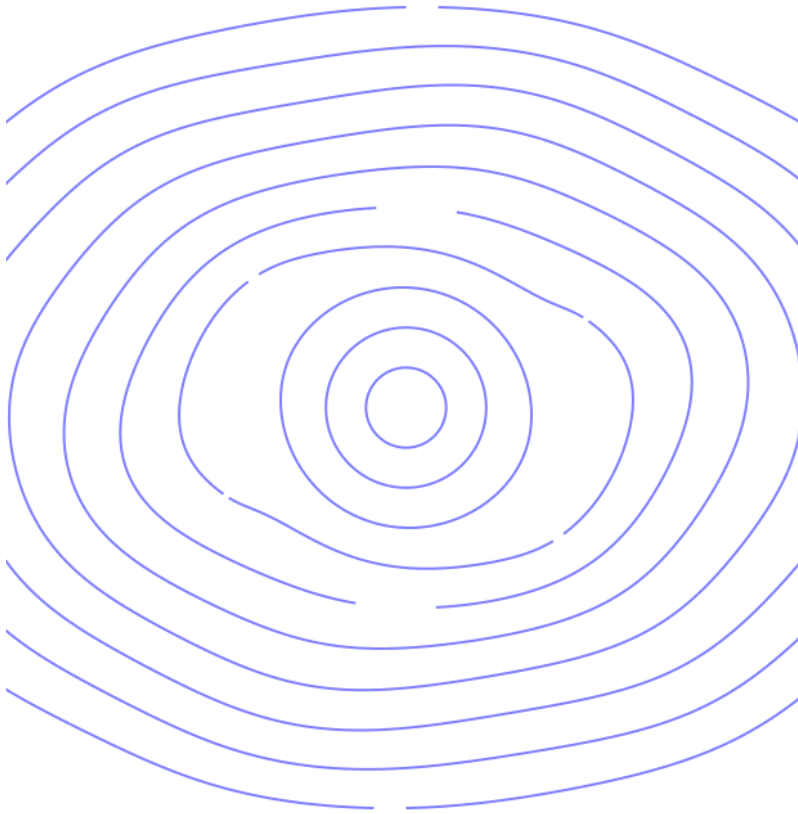


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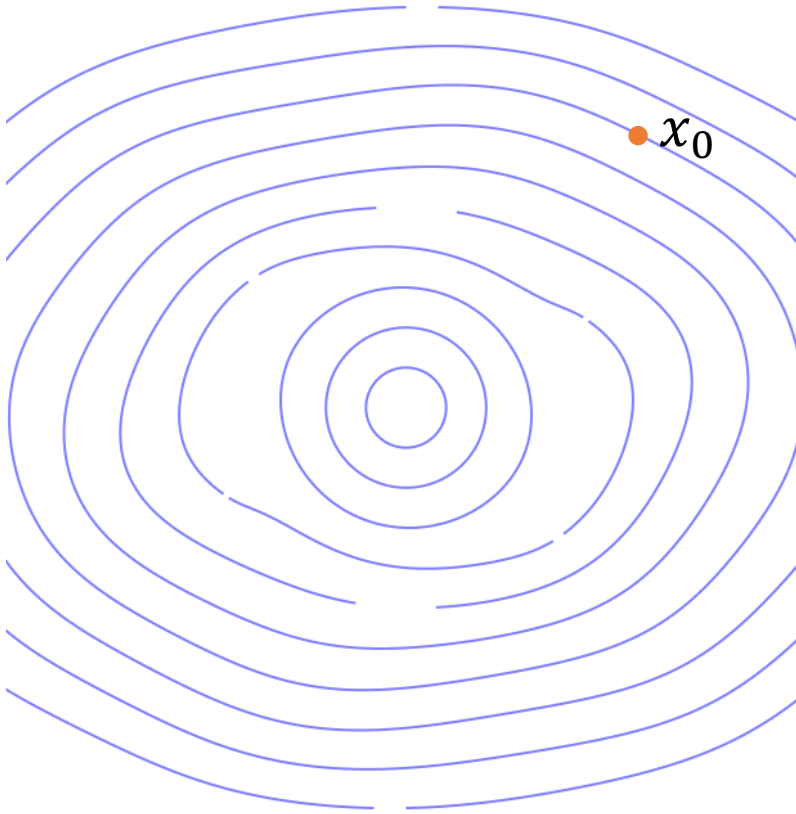


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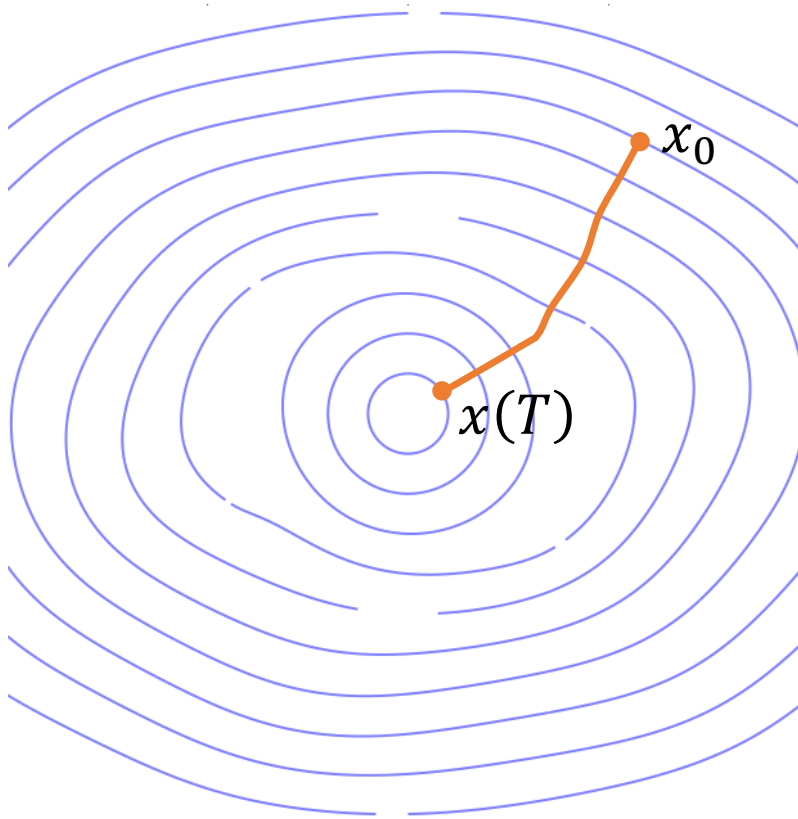
$$x_0 \mapsto \phi(x_0)$$

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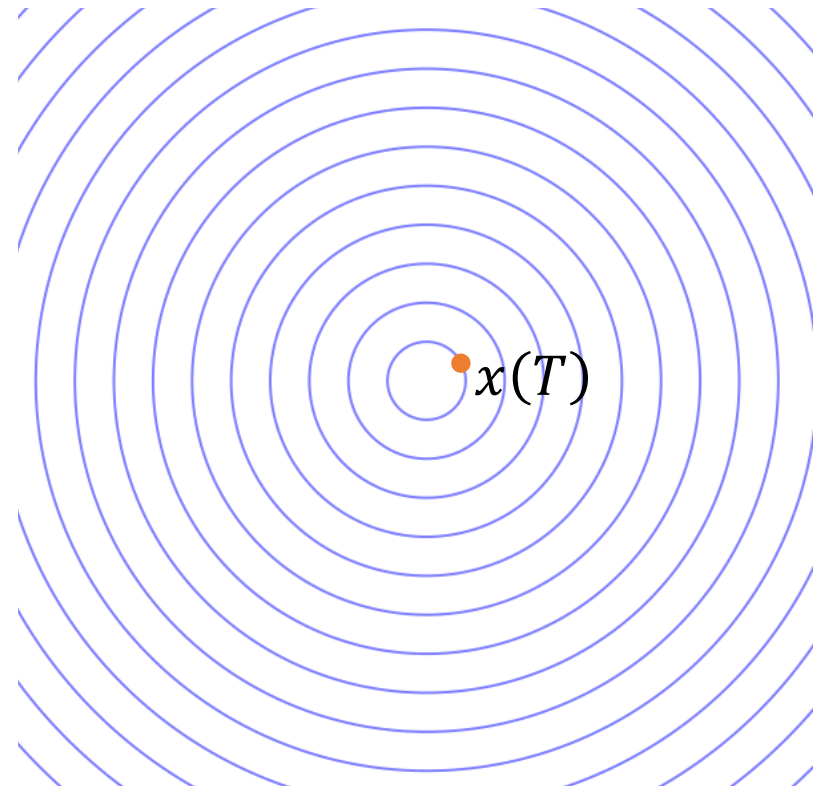
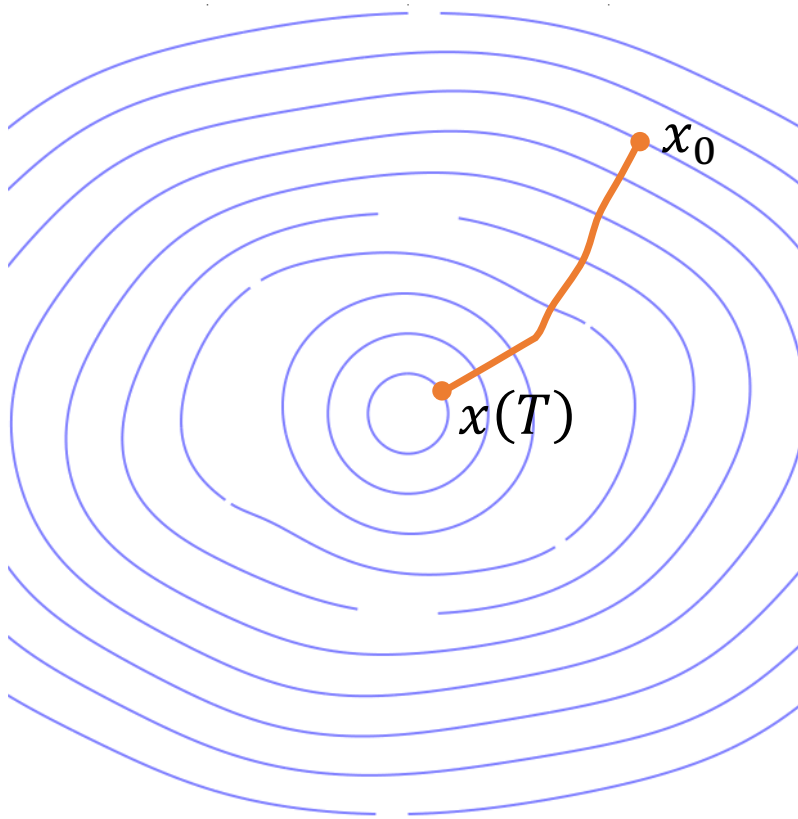
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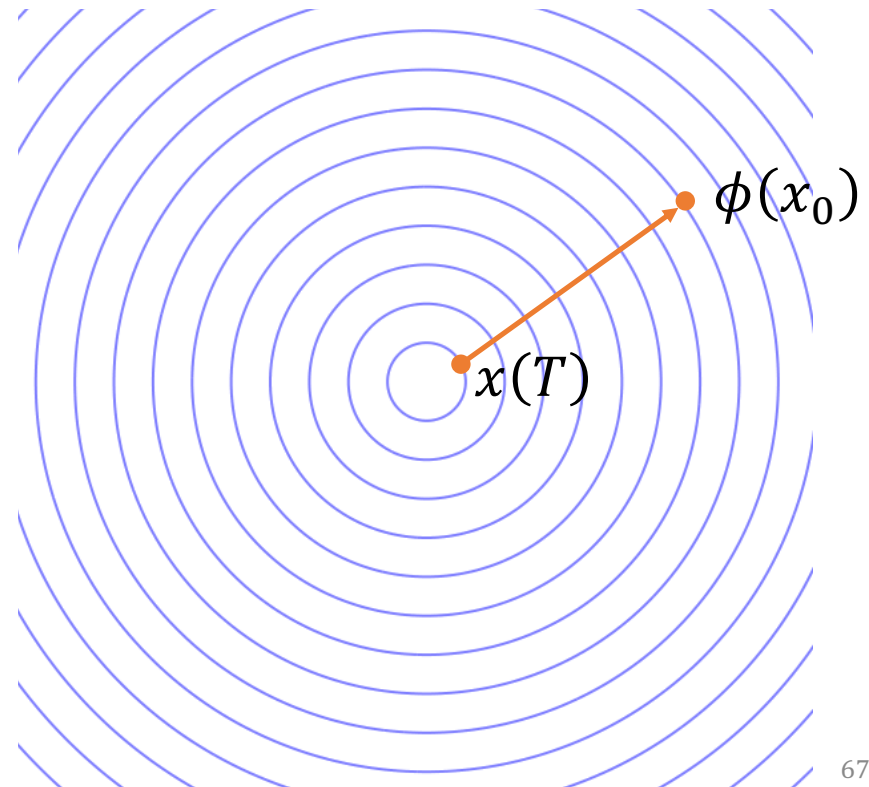
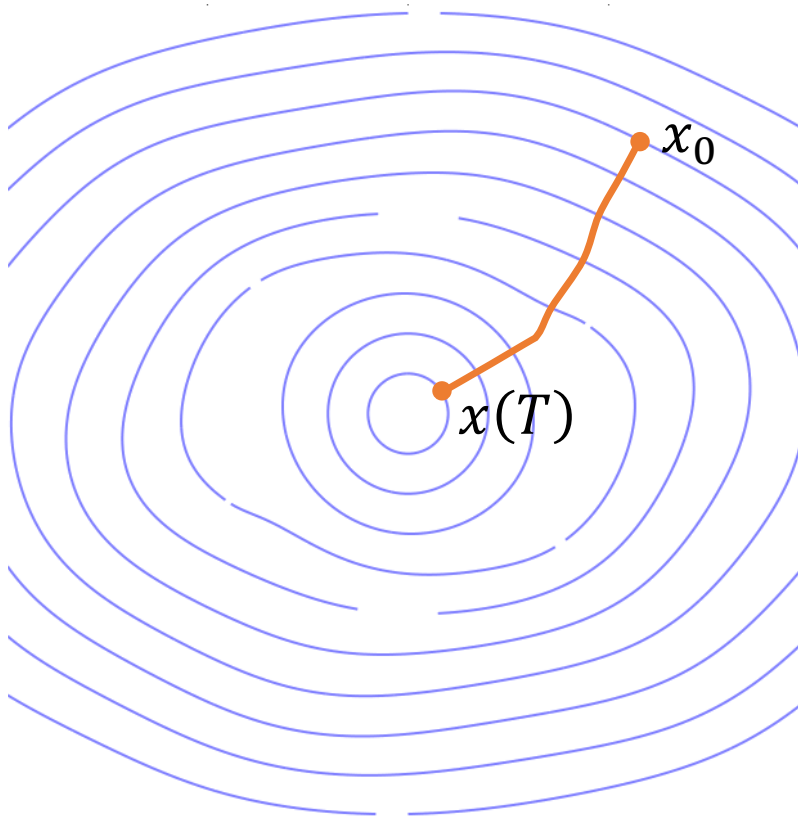


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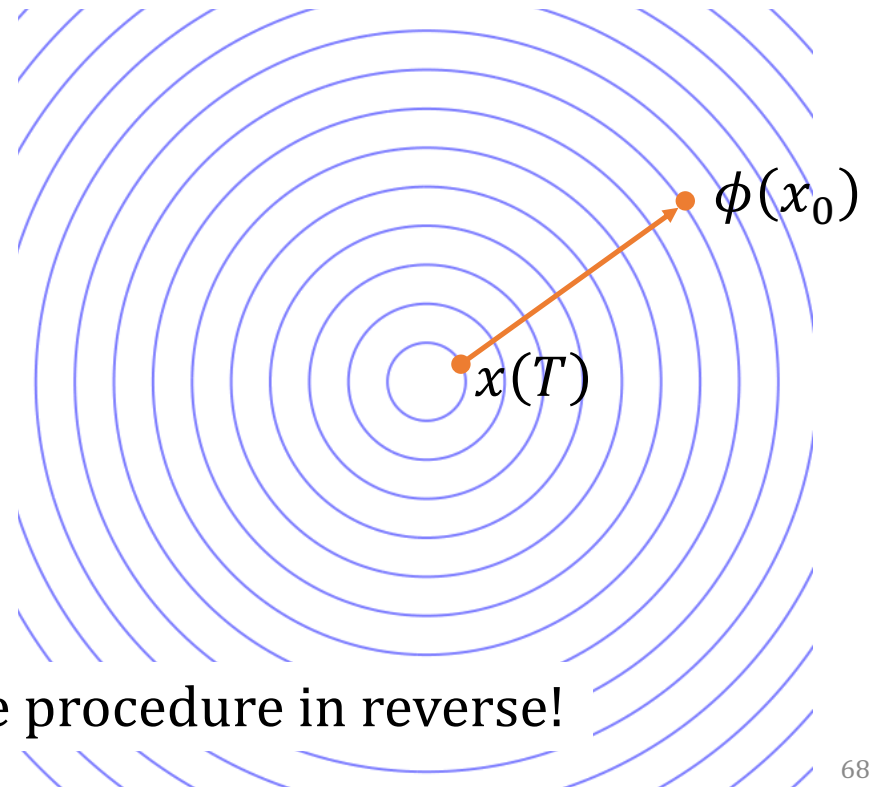
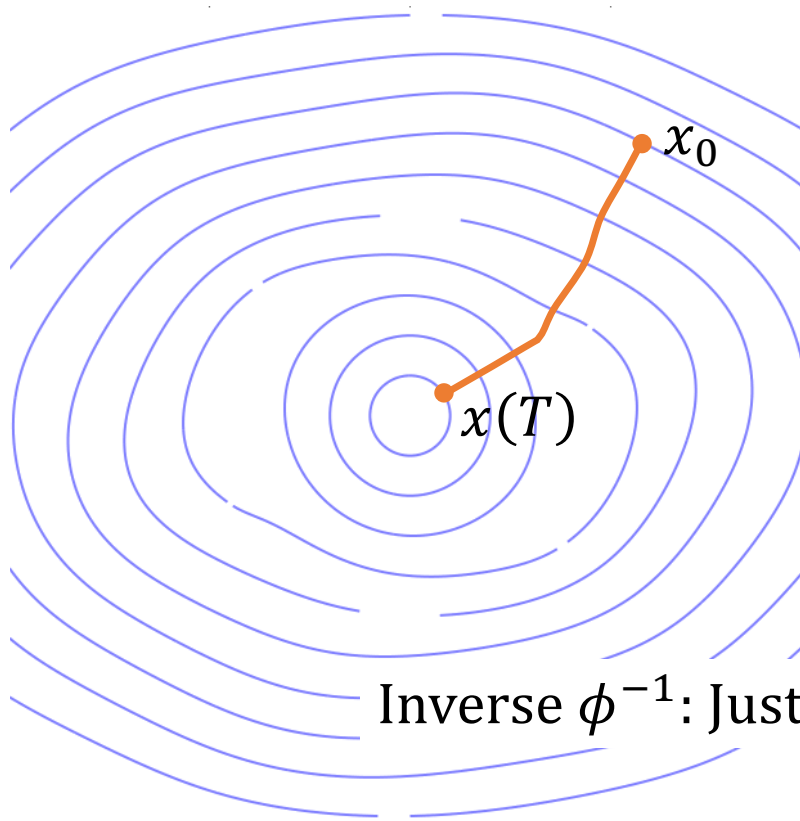


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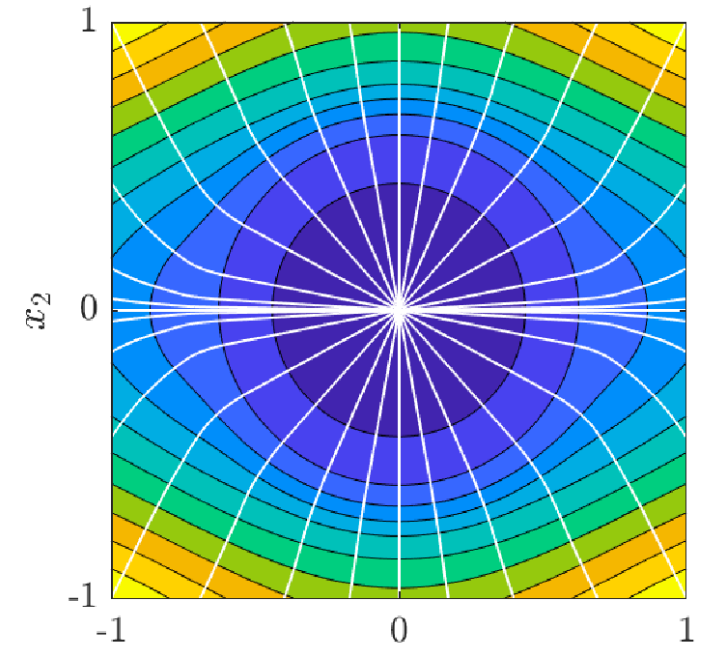
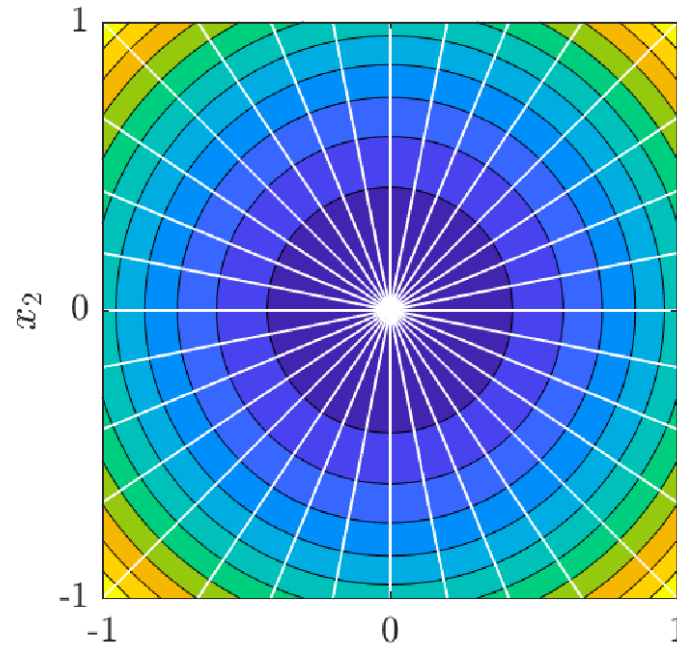
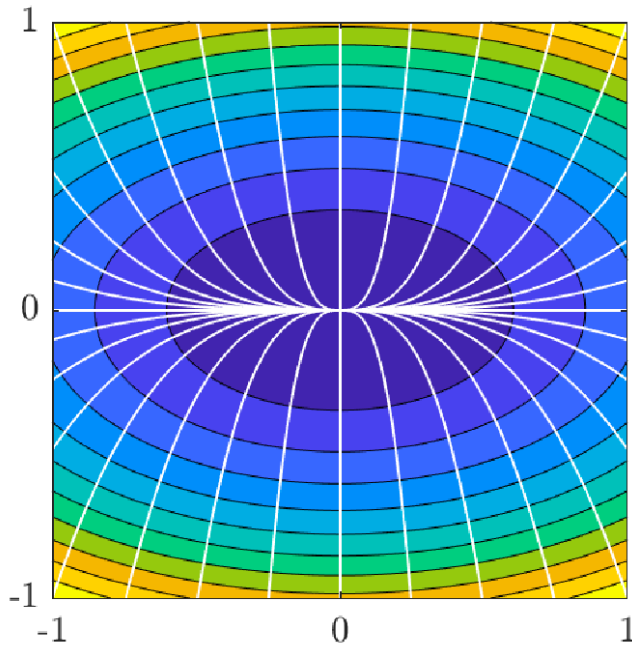
Inverse ϕ^{-1} : Just run the procedure in reverse!

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It is important to first perform Step 1, **as o.w. grad flow lines collapse.**



Remainder of talk

Baby Thm: If f is smooth and globally PL with **unique minimizer**, there exists a **diffeomorphism** $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

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Next: Some geometric intuition behind proofs

Manifold of minimizers

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$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*), \quad \forall x \in \mathbb{R}^n$$

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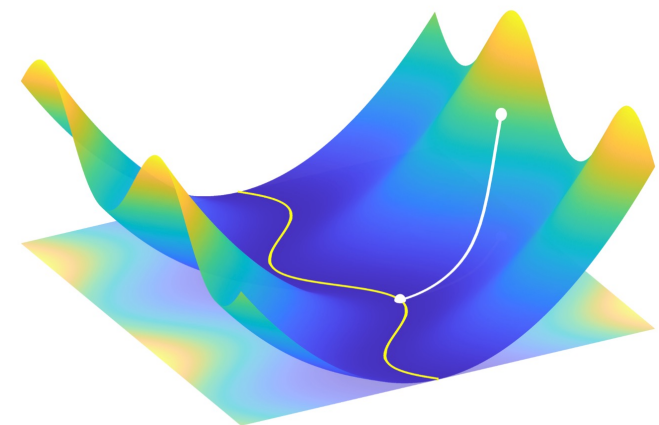
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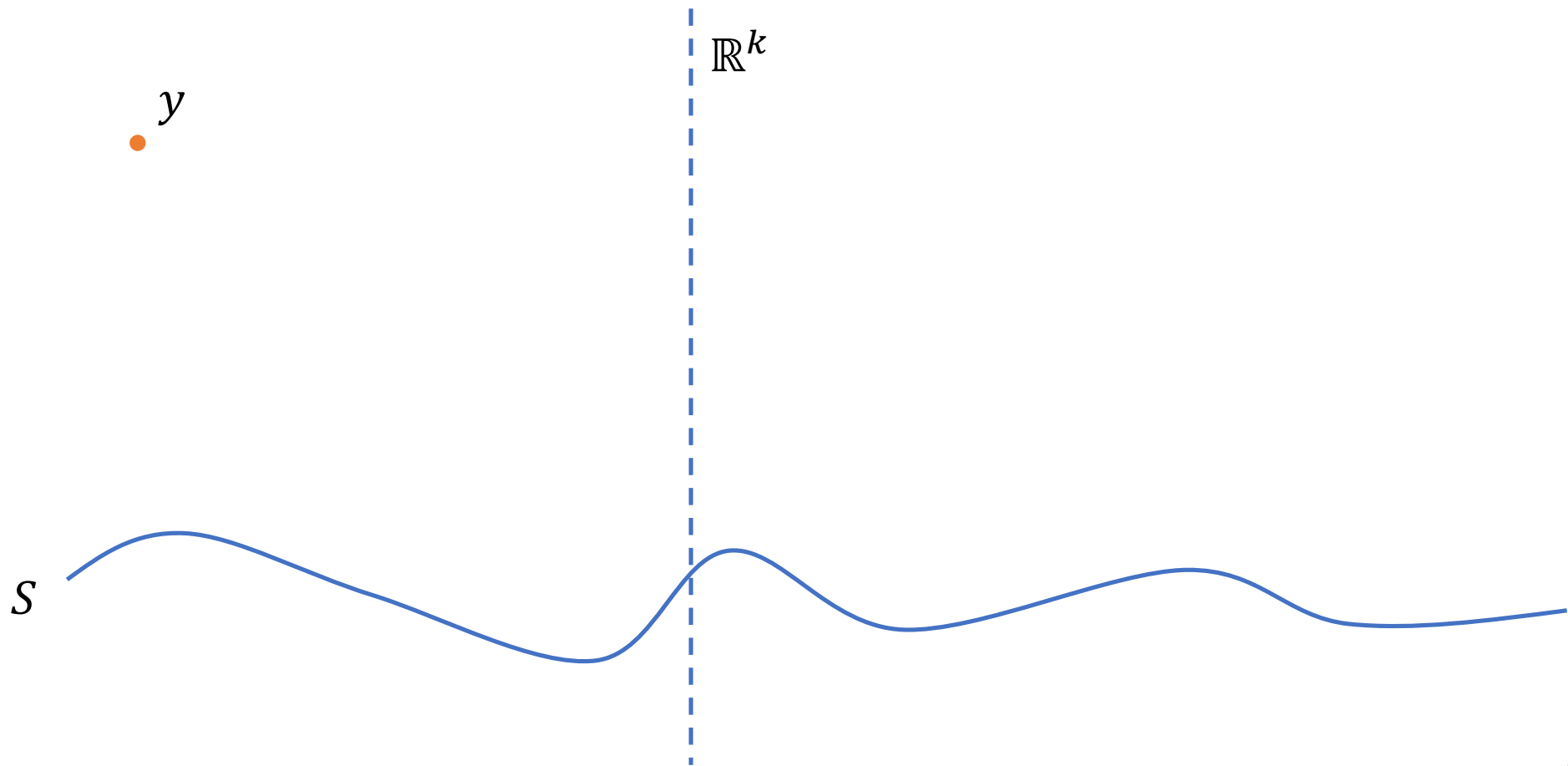


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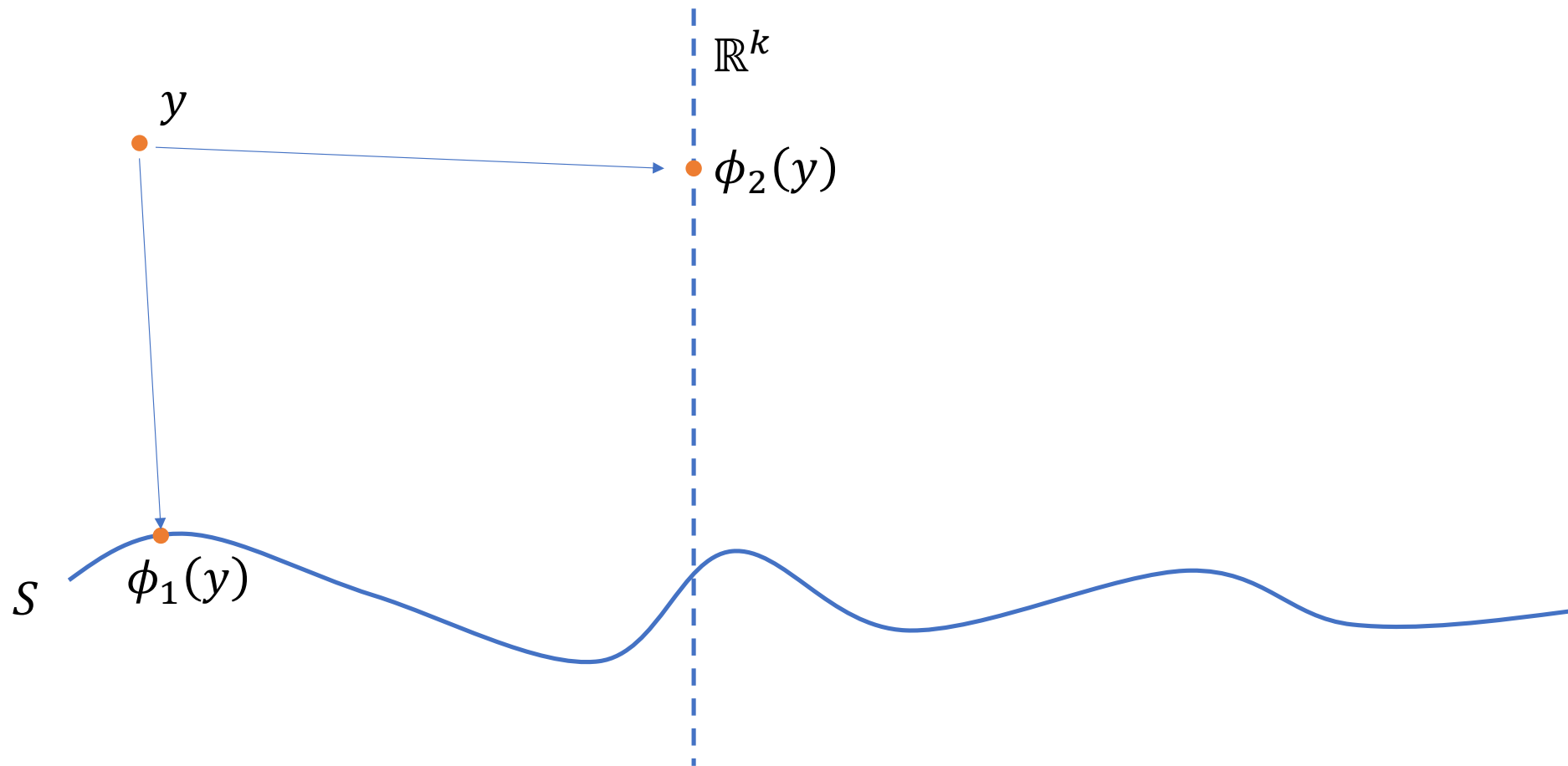


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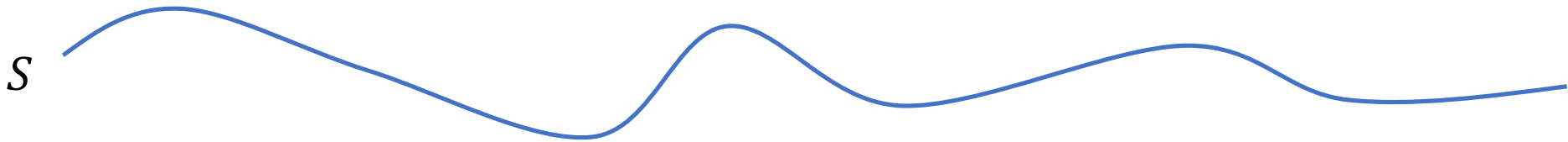
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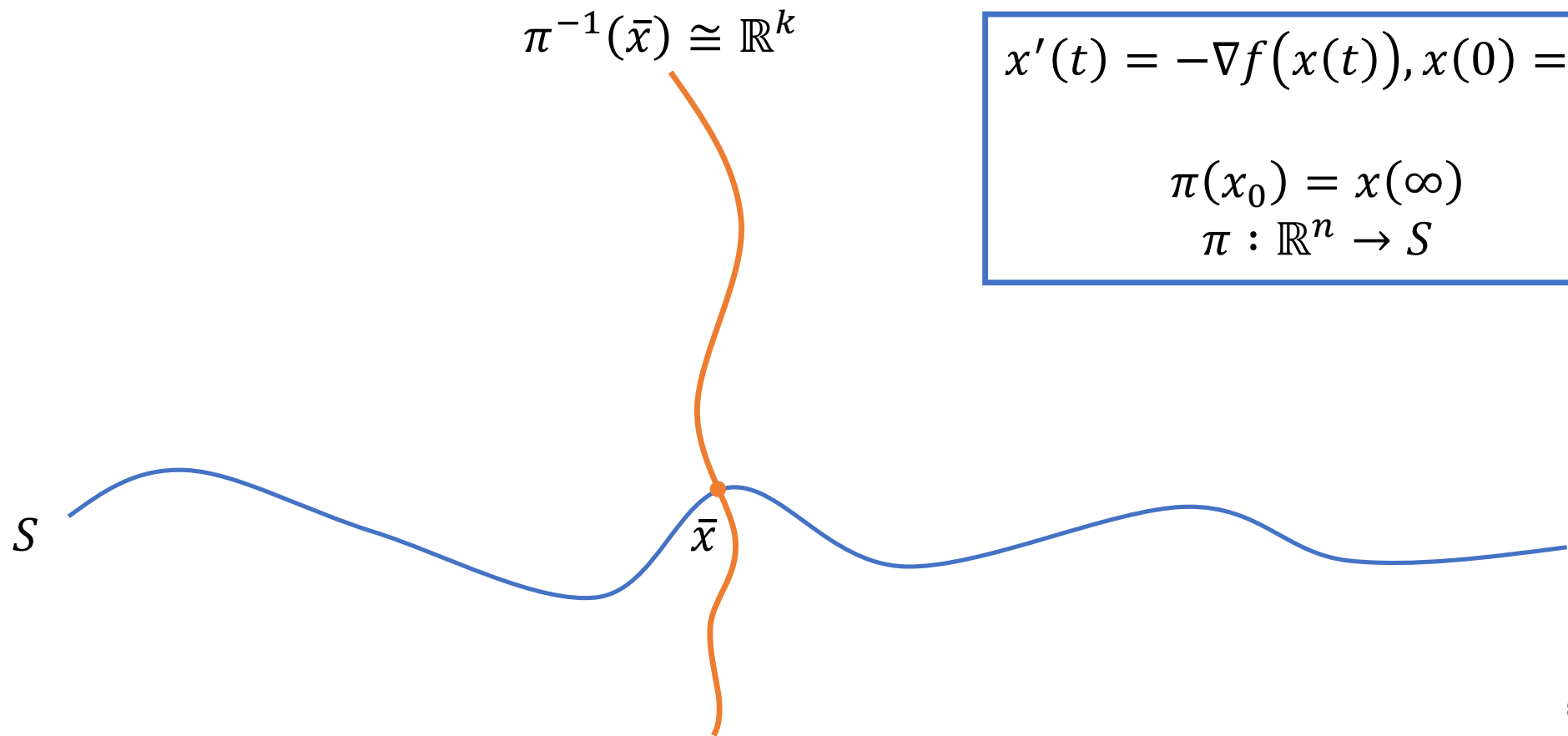


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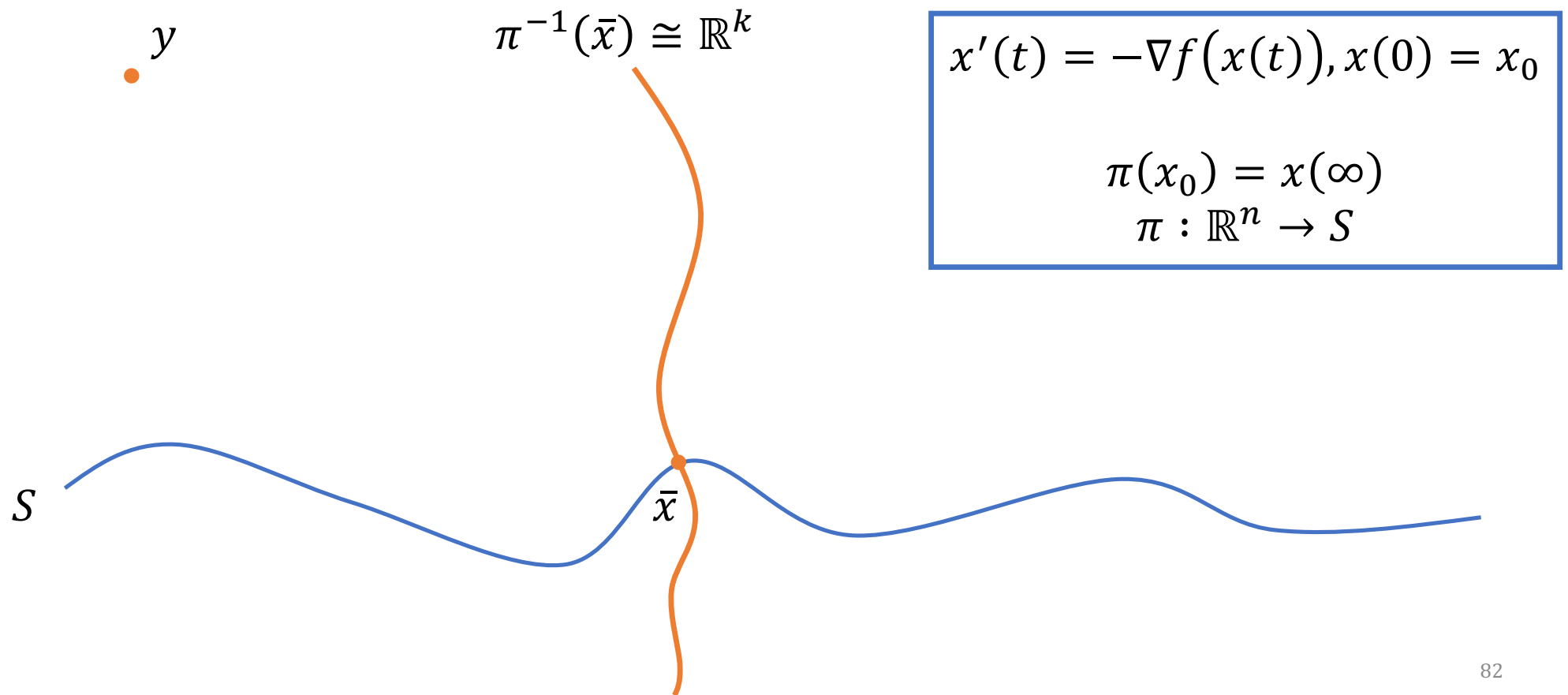
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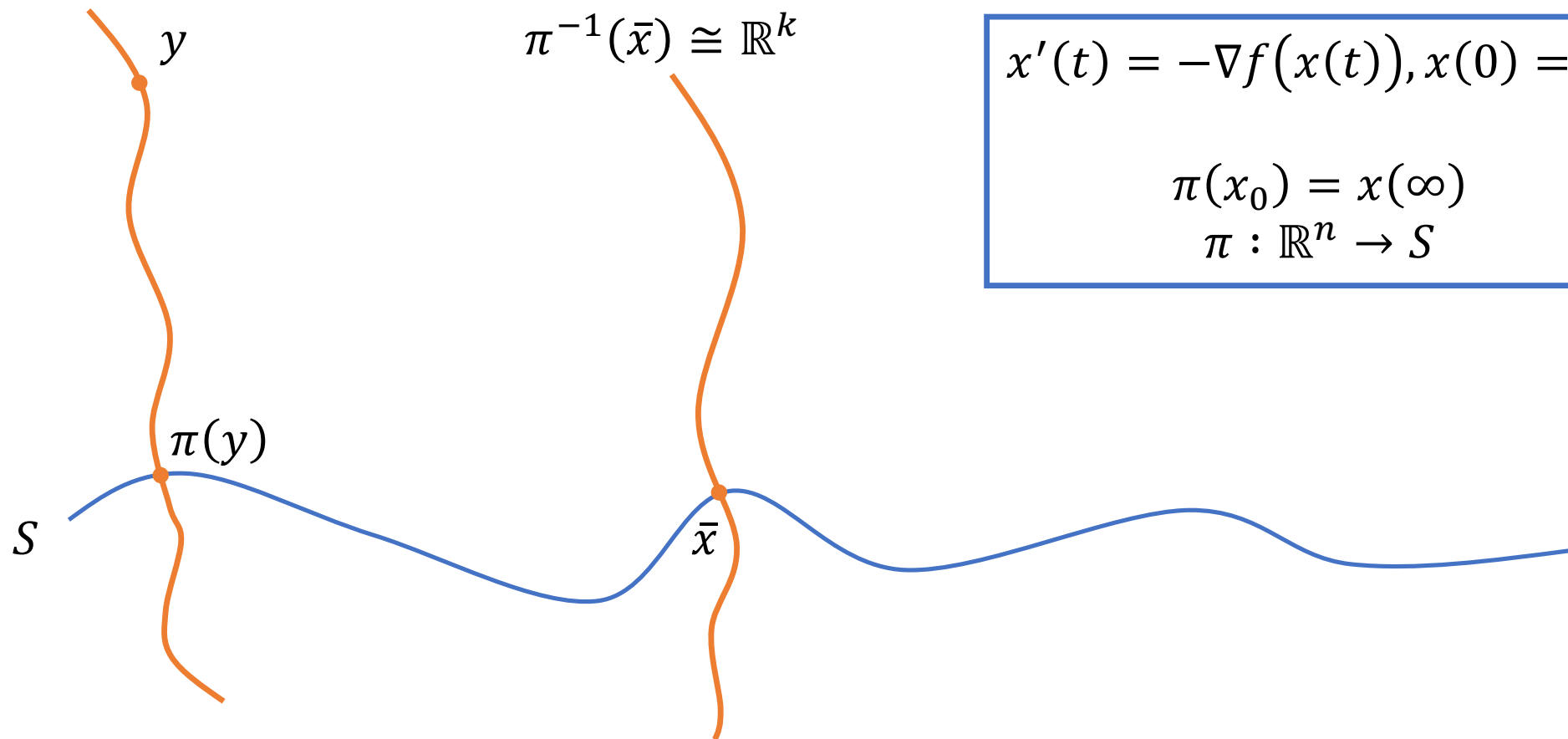


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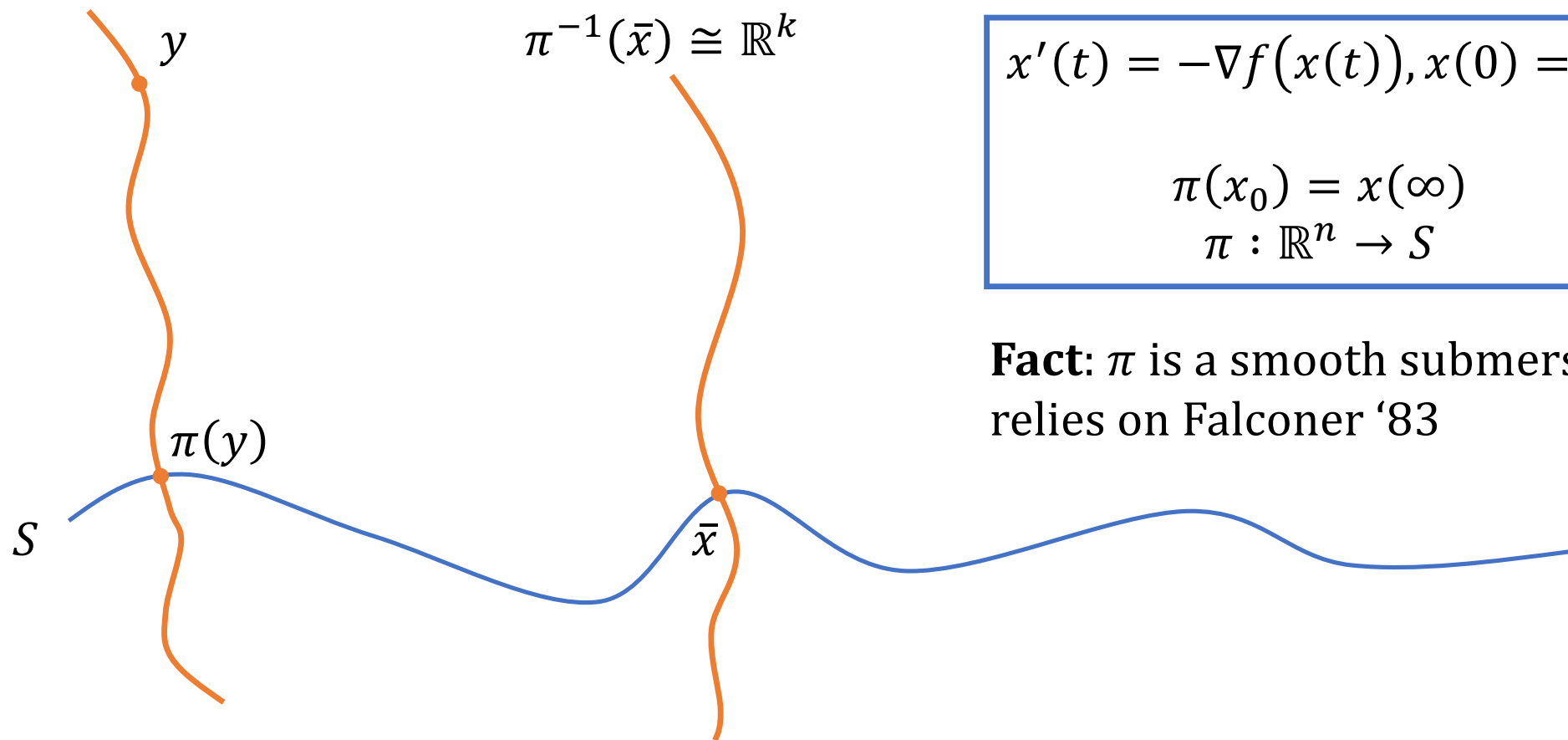
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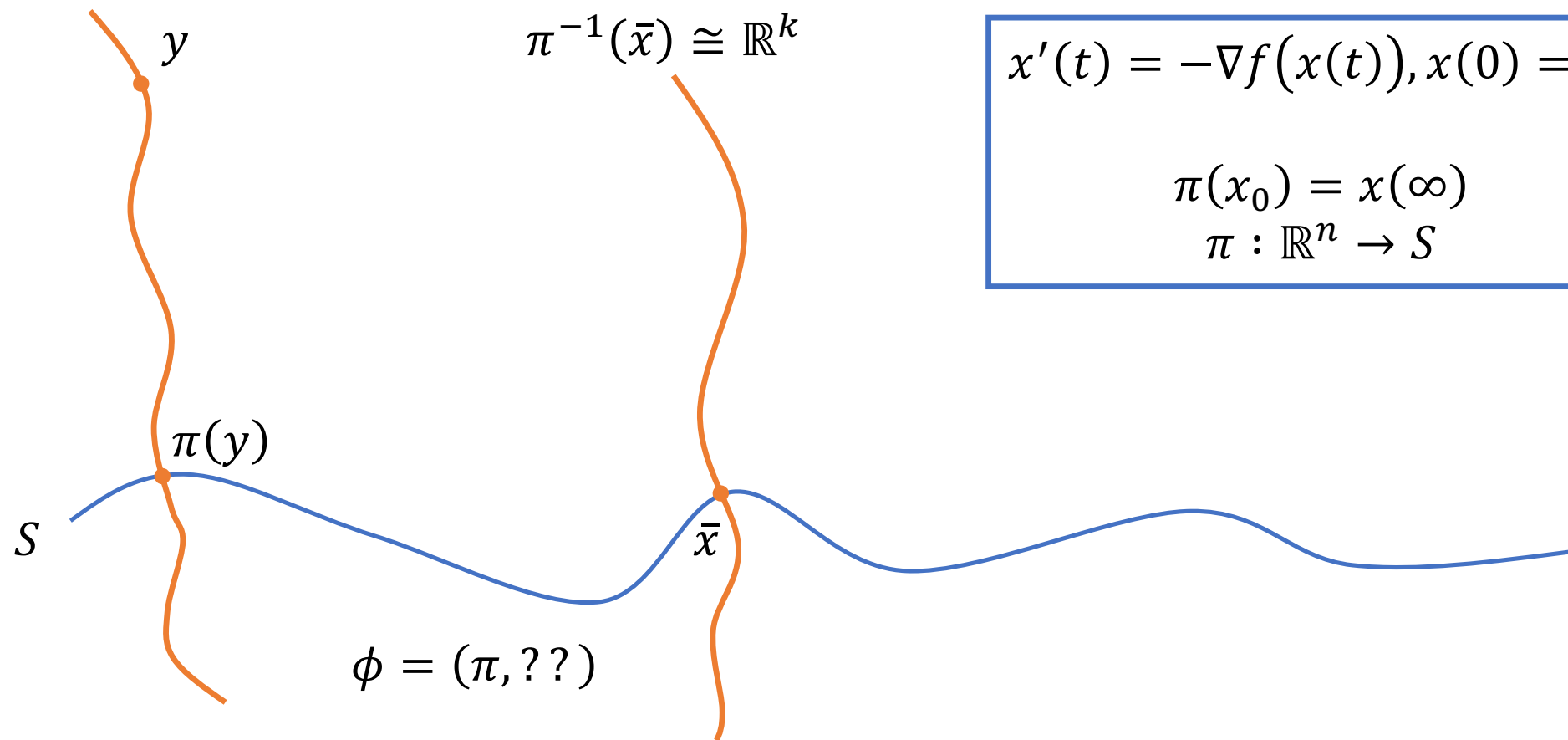
Fact: π is a smooth submersion,
relies on Falconer '83

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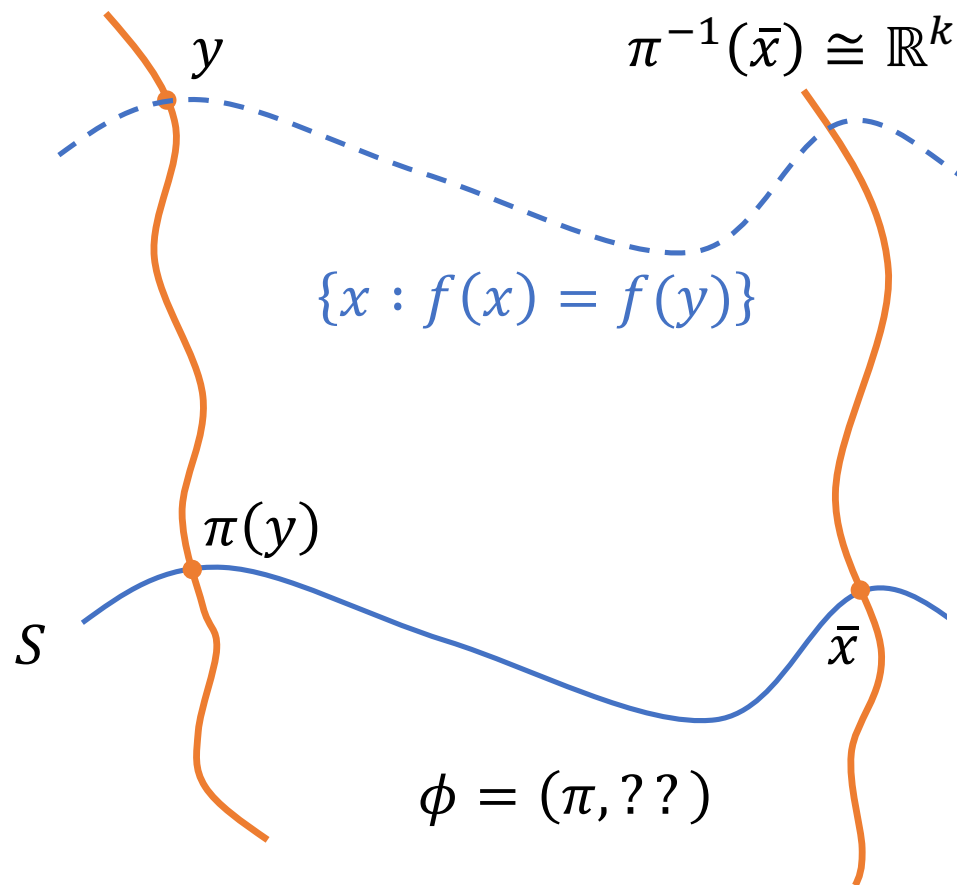


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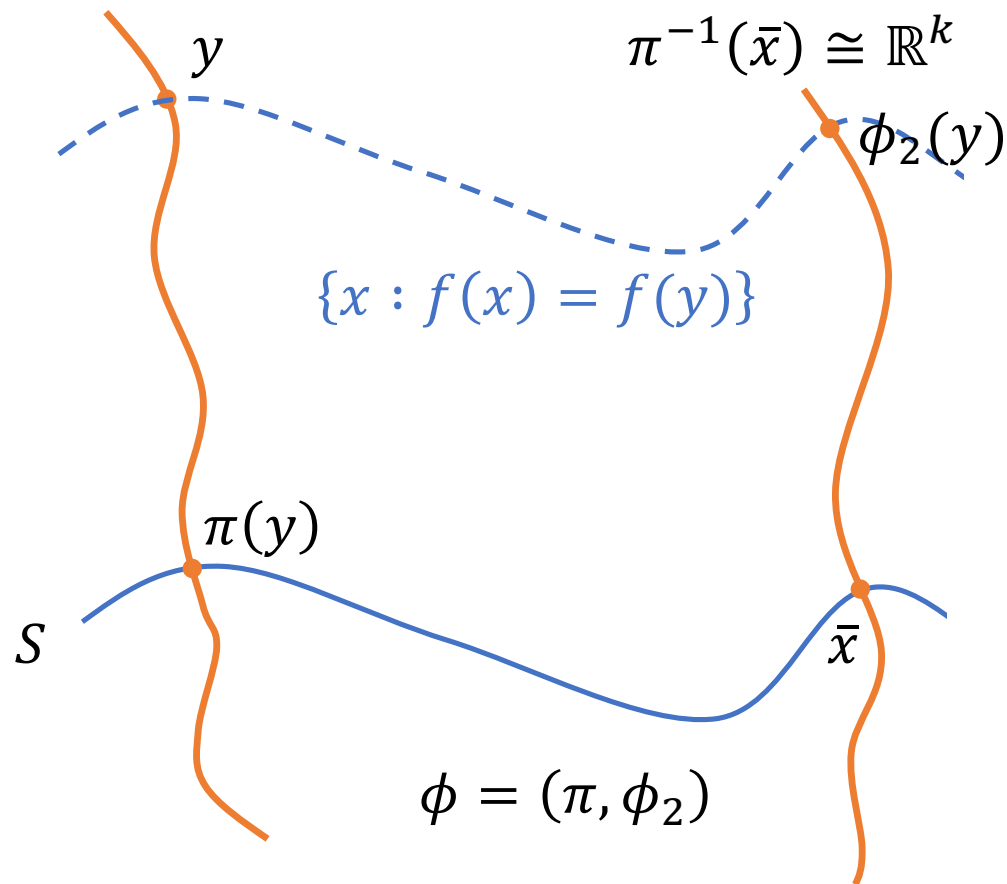
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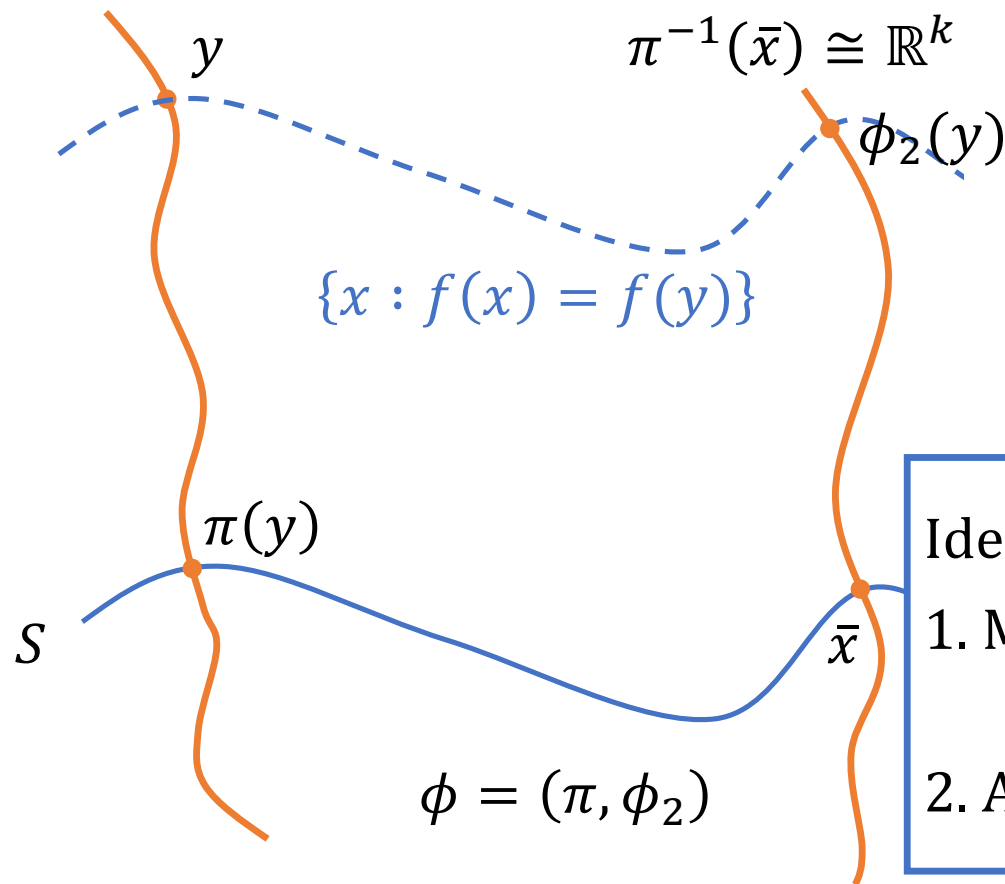
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1. Map $\phi_2 : \pi^{-1}(\pi(y)) \rightarrow \pi^{-1}(\bar{x})$, so that

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2. Apply baby thm to f restricted to $\pi^{-1}(\bar{x})$

Manifold of minimizers

$\pi^{-1}(\bar{x}) \cong \mathbb{R}^k$

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Technical terminology:

$\pi: \mathbb{R}^n \rightarrow S$ is a trivial **fiber bundle**

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Build your own crazy PL function

Thm (Building PL fcts): Let S be a submanifold of $\mathbb{R}^n$. If there exists a diffeomorphism

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Such a diffeomorphism exists for
 $S = \text{Whitehead manifold}$
[McMillan '61 and Stallings '62]

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For each $y \in \mathbb{R}^n$, let $c_y(t) = \phi^{-1}(\phi_1(y), t \phi_2(y))$.

This a curve from some point in S ($t = 0$) to y ($t = 1$).

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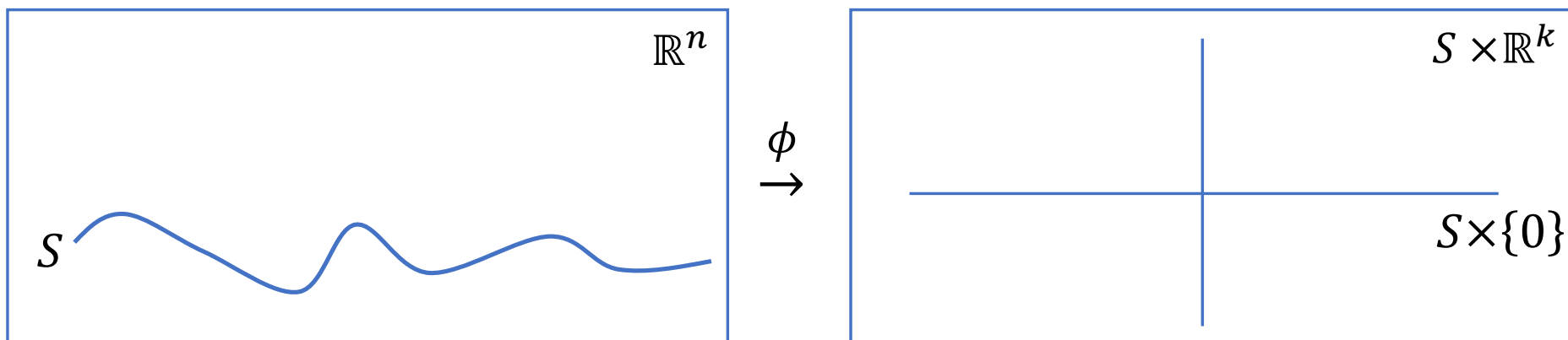
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Thm: If f is smooth, globally PL and non-constant on contractible M , then M is diffeomorphic to $\mathbb{R}^n$.